

Smooth solutions of the Euler and Navier-Stokes equations from the a posteriori analysis of approximate solutions

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Abstract

The main result of (C. Morosi and L. Pizzocchero, Nonlinear Analysis, 2012 [14]) is presented in a variant, based on a C^∞ formulation of the Cauchy problem; in this approach, the a posteriori analysis of an approximate solution gives a bound on the Sobolev distance of any order between the exact and the approximate solution.

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1 Introduction

Let us consider the homogeneous incompressible Navier-Stokes (NS) equations on a torus \mathbf{T}^d of arbitrary dimension d ; these read

$$\frac{\partial u}{\partial t} = \nu \Delta u + \mathcal{P}(u, u) + f. \quad (1.1)$$

Here: $u = u(x, t)$ is the divergence free velocity field, depending on $x \in \mathbf{T}^d$ and on time t ; $\nu \geq 0$ is the kinematic viscosity, and Δ is the Laplacian of \mathbf{T}^d ; \mathcal{P} is the bilinear map sending any two sufficiently regular vector fields $v, w : \mathbf{T}^d \rightarrow \mathbf{R}^d$ into

$$\mathcal{P}(v, w) := -\mathfrak{L}(v \bullet \partial w) . \quad (1.2)$$

In the above $v \bullet \partial w : \mathbf{T}^d \rightarrow \mathbf{R}^d$ is the vector field of components $(v \bullet \partial w)_r = \sum_{s=1}^d v_s \partial_s w_r$, and \mathfrak{L} is the Leray projection onto the space of divergence free vector fields. Finally, Eq. (1.1) contains the (Leray projected) density $f = f(x, t)$ of the external forces. In the inviscid case $\nu = 0$, the NS equations become the Euler equations.

In our previous work [14], exact and approximate solutions of the NS Cauchy problem have been discussed in a framework based on the Sobolev spaces $\mathbb{H}_{\Sigma_0}^n$, for appropriate n . For each real n , $\mathbb{H}_{\Sigma_0}^n$ consists of the (distributional) vector fields $v : \mathbf{T}^d \rightarrow \mathbf{R}^d$ with vanishing divergence and mean such that $(-\Delta)^{n/2}v$ is square integrable; this space carries the inner product $\langle v | w \rangle_n := \langle (-\Delta)^{n/2}v | (-\Delta)^{n/2}w \rangle_{L^2}$ and the corresponding norm $\| \cdot \|_n$ (see the forthcoming Eqs. (2.1)(2.2)). After fixing an order $n > d/2 + 1$, a forcing f and an initial datum $u_0 \in \mathbb{H}_{\Sigma_0}^{n+2}$, in the cited work we have discussed exact and approximate solutions of the NS Cauchy problem in the functional class $C([0, T], \mathbb{H}_{\Sigma_0}^{n+2}) \cap C^1([0, T], \mathbb{H}_{\Sigma_0}^n)$ (with T possibly depending on the solution; in the Euler case $\nu = 0$, one can harmlessly replace $\mathbb{H}_{\Sigma_0}^{n+2}$ with $\mathbb{H}_{\Sigma_0}^{n+1}$). The limitation $n > d/2 + 1$ was motivated by the basic features of the bilinear map \mathcal{P} : this sends continuously $\mathbb{H}_{\Sigma_0}^n \times \mathbb{H}_{\Sigma_0}^{n+1}$ into $\mathbb{H}_{\Sigma_0}^n$ whenever $n > d/2$ and fulfills the known Kato inequality, essential for our purposes, if $n > d/2 + 1$ (see Section 2; the Kato inequality reviewed therein reads $|\langle \mathcal{P}(v, w) | w \rangle_n| \leq G_n \|v\|_n \|w\|_n^2$, with G_n a suitable constant, also depending on d).

The method proposed in [14] was inspired by Chernyshenko *et al.* [2] (and, partly, by [12] [13]); given an approximate solution $u_a \in C([0, T_a], \mathbb{H}_{\Sigma_0}^{n+2}) \cap C^1([0, T_a], \mathbb{H}_{\Sigma_0}^n)$ of the NS Cauchy problem, it allows to infer a lower bound on the interval of existence of the exact NS solution u , and an upper bound on the $\mathbb{H}_{\Sigma_0}^n$ distance between $u(t)$ and $u_a(t)$. This is obtained via an *a posteriori* analysis of u_a relying on the norms

$$\left\| \left(\frac{du_a}{dt} - \nu \Delta u_a - \mathcal{P}(u_a, u_a) - f \right)(t) \right\|_n , \quad \|u_a(0) - u_0\|_n \quad (1.3)$$

$$\|u_a(t)\|_n , \quad \|u_a(t)\|_{n+1} \quad (1.4)$$

($t \in [0, T_a]$), which measure the “differential error”, the “datum error” and the “growth” of u_a . The above norms, or some upper bounds for them, determine some inequalities for an unknown function $\mathcal{R}_n \in C([0, T_c], \mathbf{R})$, that we have called the *control inequalities*; these consist of a differential inequality for \mathcal{R}_n , supplemented with an inequality for the initial value $\mathcal{R}_n(0)$. The main result of [14] is the following: if the control inequalities have a solution \mathcal{R}_n with domain $[0, T_c]$, then the exact solution $u \in C([0, T], \mathbb{H}_{\Sigma_0}^{n+2}) \cap C^1([0, T], \mathbb{H}_{\Sigma_0}^n)$ of the NS Cauchy problem is such that

$$T \geq T_c, \quad \|u(t) - u_a(t)\|_n \leq \mathcal{R}_n(t) \text{ on } [0, T_c]. \quad (1.5)$$

On the other hand, it is known that the NS Cauchy problem with C^∞ initial data and forcing has a C^∞ solution. Thus, it can be of interest to propose a variant of the approach of [14] where the a posteriori analysis of approximate solutions and its implications on the exact solution are presented in a C^∞ functional setting; this is the aim of the present paper. The starting points of our analysis are the following ones:

- (a) one can introduce the Fréchet space $\mathbb{H}_{\Sigma_0}^\infty$, intersection of the finite order Sobolev spaces $\mathbb{H}_{\Sigma_0}^p$ as p ranges in \mathbf{R} (or in any subset of \mathbf{R} unbounded from above, e.g., \mathbf{N}). This coincides (algebraically and topologically) with the space of the C^∞ vector fields $v : \mathbf{T}^d \rightarrow \mathbf{R}^d$ having zero divergence and mean.
- (b) The NS bilinear map \mathcal{P} fulfills known inequalities where a norm or an inner product of arbitrarily large Sobolev order p has a bound involving the Sobolev norms of order p (or $p + 1$) and of a fixed, lower order n (or $n + 1$): see the forthcoming Eqs. (2.9)(2.10). These are “tame” inequalities in the general sense used in studies on the Nash-Moser implicit function theorem [4].

Under the assumption of an initial datum $u_0 \in \mathbb{H}_{\Sigma_0}^\infty$ and of a forcing $f \in C^\infty([0, +\infty), \mathbb{H}_{\Sigma_0}^\infty)$, and given an approximate solution $u_a \in C^1([0, T_a], \mathbb{H}_{\Sigma_0}^\infty)$ of the NS Cauchy problem, the main results of the paper are as follows:

- (i) we can start from the norms (1.3)(1.4) for a given Sobolev order $n > d/2 + 1$ and reconsider the control inequalities of [14] in an unknown function $\mathcal{R}_n \in C([0, T_c], \mathbf{R})$; if these have a solution \mathcal{R}_n of domain $[0, T_c]$, then the exact solution $u \in C^\infty([0, T], \mathbb{H}_{\Sigma_0}^\infty)$ of the NS Cauchy problem fulfills the bounds (1.5). If \mathcal{R}_n is global ($T_c = +\infty$), then u is global as well ($T = +\infty$).
- (ii) For any $p > n$, the Sobolev norms

$$\left\| \left(\frac{du_a}{dt} - \nu \Delta u_a - \mathcal{P}(u_a, u_a) - f \right)(t) \right\|_p, \quad \|u_a(0) - u_0\|_p, \quad (1.6)$$

$$\|u_a(t)\|_p, \quad \|u_a(t)\|_{p+1} \quad (1.7)$$

and the function \mathcal{R}_n of item (i) can be used to construct linear inequalities for an unknown real function on $[0, T_c)$; these have an explicitly computable solution $\mathcal{R}_p : [0, T_c) \rightarrow \mathbf{R}$, and we prove that

$$\|u(t) - u_{\mathbf{a}}(t)\|_p \leq \mathcal{R}_p(t) \quad (1.8)$$

for all $t \in [0, T_c)$.

In a few words: a suitable a posteriori analysis of $u_{\mathbf{a}}$ gives bounds on the exact NS solution u in the Sobolev norms of arbitrary order.

The simplest application of the above scheme is set up choosing $u_{\mathbf{a}}(t) := 0$ for all $t \geq 0$. With this choice (and assuming $f = 0$ as a further simplification), we can obtain from the control inequalities simple and fully explicit bounds on the NS solution u and its time of existence T ; these imply, for example, that u is global and exponentially decaying in all Sobolev norms if the datum is sufficiently small, to be precise if $\|u_0\|_n \leq \nu/G_n$ (a result which is not at all surprising but might have some interest in its present, fully quantitative formulation). As a matter of fact, the general scheme (i)(ii) outlined before has been mainly devised for more sophisticated applications; however, these results will just be sketched in the present paper.

The paper is organized as follows. Section 2 and the related Appendix A are devoted to some preliminaries; in particular, they describe the inequalities for the NS bilinear map \mathcal{P} which have been mentioned in the previous discussion. Section 3 and the related Appendix B deal with the NS Cauchy problem in an $\mathbb{H}_{\Sigma_0}^\infty$ framework. Section 4 contains the main result of the paper (Proposition 4.4), corresponding to the previous items (i)(ii). Section 5 applies this results with the simple choice $u_{\mathbf{a}}(t) := 0$ for all t (assuming $f = 0$ for simplicity). Section 6 indicates the possibility of more sophisticated choices of $u_{\mathbf{a}}$, reconsidering from the viewpoint of the present work some applications of the general method of [14] presented therein and in some related works [17] [18] [19] [20]. In these applications $u_{\mathbf{a}}$ was a Galerkin approximant, or a truncated expansion in the Reynolds number or in time, typically in dimension $d = 3$; here we only indicate how these applications could be refined along the scheme of the present paper, leaving the details to future works.

2 Preliminaries

Function spaces of vector fields on the torus. Throughout this paper we work on a torus $\mathbf{T}^d := (\mathbf{R}/2\pi\mathbf{Z})^d$ of any dimension $d \in \{2, 3, \dots\}$, keeping all the notations employed in [14] (and in most of the other works of ours, cited in the bibliography). In particular, we write $\mathbb{D}'(\mathbf{T}^d) \equiv \mathbb{D}'$ for the space of \mathbf{R}^d -valued distributions on \mathbf{T}^d ; each $v \in \mathbb{D}'$ has a weakly convergent Fourier expansion $v = (2\pi)^{-d/2} \sum_{k \in \mathbf{Z}^d} v_k e^{ik \bullet x}$, with coefficients $v_k = \overline{v_{-k}} \in \mathbf{C}^d$. The mean value $\langle v \rangle$ is, by definition, the action of v on the constant test function $(2\pi)^{-d}$, and $\langle v \rangle = (2\pi)^{-d/2} v_0$. The Laplacian of $v \in \mathbb{D}'$ has Fourier coefficients $(\Delta v)_k := -|k|^2 v_k$; if $\langle v \rangle = 0$ and $n \in \mathbf{R}$, we define $(-\Delta)^{n/2} v$ to be the element of \mathbb{D}' with mean zero and Fourier coefficients $((-\Delta)^{n/2} v)_k = |k|^n v_k$ for $k \in \mathbf{Z}^d \setminus \{0\}$.

Let us consider the space $L^2(\mathbf{T}^d, \mathbf{R}^d) \equiv \mathbb{L}^2$, with its standard inner product $\langle \cdot | \cdot \rangle_{L^2}$. For any $n \in \mathbf{R}$, we consider the Sobolev space

$$\begin{aligned} \mathbb{H}_{\Sigma_0}^n(\mathbf{T}^d) &\equiv \mathbb{H}_{\Sigma_0}^n := \{v \in \mathbb{D}' \mid \operatorname{div} v = 0, \langle v \rangle = 0, (-\Delta)^{n/2} v \in \mathbb{L}^2\} \\ &= \{v \in \mathbb{D}' \mid k \bullet v_k = 0 \ \forall k \in \mathbf{Z}^d, \ v_0 = 0, \sum_{k \in \mathbf{Z}^d \setminus \{0\}} |k|^{2n} |v_k|^2 < +\infty\} \end{aligned} \quad (2.1)$$

(the lowercase symbols Σ, \circ are used to recall the vanishing of the divergence and of the mean). The above Sobolev space is equipped with the inner product and with the induced norm

$$\langle v | w \rangle_n := \langle (-\Delta)^{n/2} v | (-\Delta)^{n/2} w \rangle_{L^2} = \sum_{k \in \mathbf{Z}^d \setminus \{0\}} |k|^{2n} \overline{v_k} \bullet w_k, \quad \|v\|_n := \sqrt{\langle v | v \rangle_n}. \quad (2.2)$$

One has $\mathbb{H}_{\Sigma_0}^p \hookrightarrow \mathbb{H}_{\Sigma_0}^n$ if $p \geq n$, where \hookrightarrow indicates a continuous imbedding (more quantitatively: $\| \cdot \|_p \geq \| \cdot \|_n$ if $p \geq n$). The vector space

$$\mathbb{H}_{\Sigma_0}^\infty := \bigcap_{p \in \mathbf{R}} \mathbb{H}_{\Sigma_0}^p \quad (2.3)$$

can be equipped with the topology induced by the family of all Sobolev norms $\| \cdot \|_p$ ($p \in \mathbf{R}$). This space and its topology do not change if \mathbf{R} is replaced with any subset of the reals unbounded from above, e.g., \mathbf{N} ; the countability of the family of norms $\| \cdot \|_p$ ($p \in \mathbf{N}$) ensures that we have a Fréchet topology.

For $k \in \mathbf{N} \cup \{\infty\}$ we consider the space

$$\mathbb{C}_{\Sigma_0}^k(\mathbf{T}^d) \equiv \mathbb{C}_{\Sigma_0}^k := \{v \in C^k(\mathbf{T}^d, \mathbf{R}^d) \mid \operatorname{div} v = 0, \langle v \rangle = 0\}, \quad (2.4)$$

which is a Banach space for $k < \infty$ and a Fréchet space for $k = \infty$, when equipped with the sup norms for all derivatives up to order k . Let $h, k \in \mathbf{N}$, $p \in \mathbf{R}$; then $\mathbb{C}_{\Sigma_0}^h \hookrightarrow \mathbb{H}_{\Sigma_0}^p$ if $h \geq p$ and, by the Sobolev lemma, $\mathbb{H}_{\Sigma_0}^p \hookrightarrow \mathbb{C}_{\Sigma_0}^k$ if $p > k + d/2$. From these facts one easily infers

$$\mathbb{H}_{\Sigma_0}^\infty = \mathbb{C}_{\Sigma_0}^\infty \quad (2.5)$$

(which indicates the equality of the above vector spaces and of their Fréchet topologies).

The NS bilinear map, and some inequalities for it. We have already introduced the notation \mathcal{P} for the fundamental bilinear map in the NS equations on \mathbf{T}^d , see Eq. (1.2).

Let n, p be real numbers fulfilling the inequalities written hereafter; it is known that

$$n > d/2, v \in \mathbb{H}_{\Sigma_0}^n, w \in \mathbb{H}_{\Sigma_0}^{n+1} \Rightarrow \mathcal{P}(v, w) \in \mathbb{H}_{\Sigma_0}^n \quad (2.6)$$

and that there are constants $K_n, G_n, K_{pn}, G_{pn} \in (0, +\infty)$ such that the following holds:

$$\|\mathcal{P}(v, w)\|_n \leq K_n \|v\|_n \|w\|_{n+1} \quad \text{for } n > d/2, v \in \mathbb{H}_{\Sigma_0}^n, w \in \mathbb{H}_{\Sigma_0}^{n+1}, \quad (2.7)$$

$$|\langle \mathcal{P}(v, w) | w \rangle_n| \leq G_n \|v\|_n \|w\|_n^2 \quad \text{for } n > d/2 + 1, v \in \mathbb{H}_{\Sigma_0}^n, w \in \mathbb{H}_{\Sigma_0}^{n+1}, \quad (2.8)$$

$$\|\mathcal{P}(v, w)\|_p \leq \frac{1}{2} K_{pn} (\|v\|_p \|w\|_{n+1} + \|v\|_n \|w\|_{p+1}) \quad (2.9)$$

$$\text{for } p \geq n > d/2, v \in \mathbb{H}_{\Sigma_0}^p, w \in \mathbb{H}_{\Sigma_0}^{p+1},$$

$$|\langle \mathcal{P}(v, w) | w \rangle_p| \leq \frac{1}{2} G_{pn} (\|v\|_p \|w\|_n + \|v\|_n \|w\|_p) \|w\|_p \quad (2.10)$$

$$\text{for } p \geq n > d/2 + 1, v \in \mathbb{H}_{\Sigma_0}^p, w \in \mathbb{H}_{\Sigma_0}^{p+1}.$$

Note that (2.9) with $p = n$ implies (2.7), with $K_n := K_{nn}$; similarly, (2.10) with $p = n$ gives (2.8) with $G_n := G_{nn}$. Statements (2.6) (2.7) indicate that \mathcal{P} maps continuously $\mathbb{H}_{\Sigma_0}^n \times \mathbb{H}_{\Sigma_0}^{n+1}$ to $\mathbb{H}_{\Sigma_0}^n$. The fact that these statements hold for all $n > d/2$ also ensures that \mathcal{P} maps continuously $\mathbb{H}_{\Sigma_0}^\infty \times \mathbb{H}_{\Sigma_0}^\infty$ to $\mathbb{H}_{\Sigma_0}^\infty$.

Eq. (2.7) is closely related to the basic norm inequalities about multiplication in Sobolev space, and (2.8) is due to Kato [5]; fully quantitative estimates for the constants K_n, G_n therein have been given in our papers [15] [16] where (2.7) and (2.8) are referred to, respectively, as the basic and Kato inequalities for \mathcal{P} . Eqs. (2.9) (2.10) could be referred to as the generalized basic and Kato inequalities; as mentioned in the Introduction, they are “tame” refinements (in the Nash-Moser sense) of Eqs. (2.7) (2.8). We remark that inequalities very similar to (2.10) are used by Temam in [24] and by Beale-Kato-Majda in [1]; recently, some analogous inequalities have been proposed by Robinson-Sadowski-Silva [23] as a tool to investigate the putative blow-up of NS solutions. Explicit expressions for the constants K_{pn}, G_{pn} in (2.9) (2.10) are given in Appendix A and in [21].

3 The NS Cauchy problem in a smooth framework

We are now ready to discuss the NS Cauchy problem in the framework of the space $\mathbb{H}_{\Sigma_0}^\infty = \mathbb{C}_{\Sigma_0}^\infty$. Let us choose

$$\nu \in [0, +\infty), \quad f \in C^\infty([0, +\infty), \mathbb{H}_{\Sigma_0}^\infty), \quad u_0 \in \mathbb{H}_{\Sigma_0}^\infty. \quad (3.1)$$

3.1 Definition. *The (incompressible) NS Cauchy problem with viscosity ν , forcing f and initial datum u_0 is the following:*

$$\text{Find } u \in C^\infty([0, T), \mathbb{H}_{\Sigma_0}^\infty) \text{ such that } \frac{du}{dt} = \nu \Delta u + \mathcal{P}(u, u) + f, \quad u(0) = u_0 \quad (3.2)$$

(with $T \in (0, +\infty]$, depending on u). If $\nu = 0$, this will also be called the “Euler Cauchy problem” with datum u_0 and forcing f .

3.2 Proposition *With ν, f, u_0 as in (3.1), the following holds.*

(i) *Problem (3.2) has a unique maximal (i.e., not extendable) solution, from now on denoted by u , with a suitable domain $[0, T)$. Every solution is a restriction of the maximal one.*

(ii) *(Beale-Kato-Majda blow up criterion.) Let u, T be as before. If $T < +\infty$, then $\int_0^T dt \|\operatorname{rot} u(t)\|_{L^\infty} = +\infty$, whence $\limsup_{t \rightarrow T^-} \|\operatorname{rot} u(t)\|_{L^\infty} = +\infty$.*

(iii) *The result (ii) implies the following: if $T < +\infty$, then for each real $n > d/2 + 1$ one has $\int_0^T dt \|u(t)\|_n = +\infty$, whence $\limsup_{t \rightarrow T^-} \|u(t)\|_n = +\infty$.*

The above proposition is known; it combines results from Kato [5], Temam [24] and Beale-Kato-Majda [1] on local existence and blow up for the Euler equations that, as indicated by the authors themselves, have simple generalizations to NS equations with arbitrary viscosity; for more details, we refer to Appendix B. In the approach of this Appendix, the main reason for local existence in $\mathbb{H}_{\Sigma_0}^\infty$ is that local existence can be established in Sobolev spaces of finite but arbitrarily large order, on a time interval independent of the order; this idea was first advanced by Temam [24], on the grounds of a blow up criterion slightly weaker than the one of Beale-Kato-Majda.

For completeness we mention that, in the case $\nu > 0$, statement (ii) can be replaced by a blow up criterion of Giga [3] [8] asserting that $\int_0^T dt \|u(t)\|_{L^\infty}^2 = +\infty$, and implying $\int_0^T dt \|u(t)\|_n^2 = +\infty$ for all $n > d/2$; this is not relevant for our present purposes since the treatment that we propose for the approximate solutions, related to the Kato and generalized Kato inequalities (2.8) (2.10), relies on the Sobolev norms of orders $> d/2 + 1$.

4 Approximate solutions of the NS Cauchy problem and control inequalities

Assuming again ν, f, u_0 as in (3.1), let us stipulate what follows.

4.1 Definition. An approximate solution of the problem (3.2) is any map $u_a \in C^1([0, T_a], \mathbb{H}_{\Sigma_0}^\infty)$, with $T_a \in (0, +\infty]$. Given such a function, we stipulate (i) (ii).

(i) The differential error of u_a is

$$e(u_a) := \frac{du_a}{dt} - \nu \Delta u_a - \mathcal{P}(u_a, u_a) - f \in C([0, T_a], \mathbb{H}_{\Sigma_0}^\infty) ; \quad (4.1)$$

the datum error is

$$u_a(0) - u_0 \in \mathbb{H}_{\Sigma_0}^\infty . \quad (4.2)$$

(ii) Let $p \in \mathbf{R}$. A differential error estimator, a datum error estimator and a growth estimator of order p for u_a are a function $\epsilon_p \in C([0, T_a], [0, +\infty))$, a number $\delta_p \in [0, +\infty)$ and a function $\mathcal{D}_p \in C([0, T_a], [0, +\infty))$ such that, respectively,

$$\|e(u_a)(t)\|_p \leq \epsilon_p(t) \quad \text{for } t \in [0, T_a] , \quad (4.3)$$

$$\|u_a(0) - u_0\|_p \leq \delta_p , \quad (4.4)$$

$$\|u_a(t)\|_p \leq \mathcal{D}_p(t) \quad \text{for } t \in [0, T_a] . \quad (4.5)$$

In particular the function $\epsilon_p(t) := \|e(u_a)(t)\|_p$, the number $\delta_p := \|u_a(0) - u_0\|_p$ and the function $\mathcal{D}_p(t) := \|u_a(t)\|_p$ will be called the tautological estimators of order p for the differential error, the datum error and the growth of u_a .

We note that, according to the previous definition, an approximate solution u_a is in $\mathbb{H}_{\Sigma_0}^\infty$ and thus is C^∞ at any instant, but is only required to be C^1 in time; stronger regularity conditions, such as the assumption that u_a is C^∞ in time, are not necessary for the sequel.

The forthcoming lemma presents an estimate on the time derivative of the Sobolev distance of any order $p > d/2 + 1$ between the exact solution of the NS Cauchy problem and an approximate solution. This estimate is the basic tool to establish our main result on approximate solutions, which is contained in Proposition 4.4. Before stating the lemma and the proposition we introduce the following notations and assumptions, to be used throughout the section:

- (I) $u \in C^\infty([0, T], \mathbb{H}_{\Sigma_0}^\infty)$ is the maximal solution of the NS Cauchy problem (3.2);
- (II) $u_a \in C^1([0, T_a], \mathbb{H}_{\Sigma_0}^\infty)$ is any approximate solution of (3.2). For each $p > d/2 + 1$, ϵ_p, δ_p and \mathcal{D}_p are estimators of order p for the differential error, the datum error and the growth of u_a ;

- (III) K_n, G_n and G_{pn} are constants fulfilling the inequalities (2.7) (2.8) (2.10), for all real n and p with the limitations indicated therein;
- (IV) for each function $\varphi : [0, \tau) \rightarrow \mathbf{R}$ ($\tau \in (0, +\infty]$), we use the *right, upper Dini derivative* $\frac{d^+\varphi}{dt} : [0, \tau) \rightarrow (-\infty, +\infty]$, $t \mapsto \frac{d^+\varphi}{dt}(t) := \limsup_{h \rightarrow 0^+} \frac{\varphi(t+h) - \varphi(t)}{h}$.

4.2 Lemma. *Consider the C^1 function*

$$u - u_{\mathbf{a}} : [0, T_\star) \rightarrow \mathbb{H}_{\Sigma_0}^\infty, \quad T_\star := \min(T, T_{\mathbf{a}}). \quad (4.6)$$

For any real p , introduce the norm $\|u - u_{\mathbf{a}}\|_p : [0, T_\star) \rightarrow [0, +\infty)$, $t \mapsto \|u(t) - u_{\mathbf{a}}(t)\|_p$ (a continuous function, possibly non-differentiable where $u(t) = u_{\mathbf{a}}(t)$). If $n, p \in \mathbf{R}$ are such that $d/2 + 1 < n \leq p < +\infty$, the following holds everywhere on $[0, T_\star)$:

$$\frac{d^+}{dt} \|u - u_{\mathbf{a}}\|_p \leq -\nu \|u - u_{\mathbf{a}}\|_p \quad (4.7)$$

$$+(G_p \mathcal{D}_p + K_p \mathcal{D}_{p+1}) \|u - u_{\mathbf{a}}\|_p + G_{pn} \|u - u_{\mathbf{a}}\|_n \|u - u_{\mathbf{a}}\|_p + \epsilon_p.$$

Proof. For the sake of brevity, we put

$$w := u - u_{\mathbf{a}}. \quad (4.8)$$

The function w fulfills

$$\frac{dw}{dt} = \nu \Delta w + \mathcal{P}(u_{\mathbf{a}}, w) + \mathcal{P}(w, u_{\mathbf{a}}) + \mathcal{P}(w, w) - e(u_{\mathbf{a}}), \quad (4.9)$$

$$w(0) = u_0 - u_{\mathbf{a}}(0). \quad (4.10)$$

Let us consider the function $\|w\|_p$. In a neighborhood of any instant t such that $w(t) \neq 0$ this function is differentiable, and

$$\frac{d^+ \|w\|_p}{dt} = \frac{d \|w\|_p}{dt} = \frac{1}{2 \|w\|_p} \frac{d \|w\|_p^2}{dt} = \frac{1}{\|w\|_p} \left\langle \frac{dw}{dt} | w \right\rangle_p \quad (4.11)$$

$$= \frac{1}{\|w\|_p} \left(\nu \langle \Delta w | w \rangle_p + \langle \mathcal{P}(u_{\mathbf{a}}, w) | w \rangle_p + \langle \mathcal{P}(w, u_{\mathbf{a}}) | w \rangle_p + \langle \mathcal{P}(w, w) | w \rangle_p - \langle e(u_{\mathbf{a}}) | w \rangle_p \right).$$

On the other hand, using the Fourier representations for Δ and for $\langle | \rangle_p$, $\| \cdot \|_{p+1}$, $\| \cdot \|_p$ we easily infer

$$\langle \Delta w | w \rangle_p = -\|w\|_{p+1}^2 \leq -\|w\|_p^2; \quad (4.12)$$

moreover, using the inequalities (2.7) (2.8) (2.10) for \mathcal{P} , the Schwarz inequality for $\langle | \rangle_p$ and the relations (4.3) (4.5) for $\epsilon_p, \mathcal{D}_p, \mathcal{D}_{p+1}$ we get

$$\langle \mathcal{P}(u_{\mathbf{a}}, w) | w \rangle_p \leq G_p \|u_{\mathbf{a}}\|_p \|w\|_p^2 \leq G_p \mathcal{D}_p \|w\|_p^2, \quad (4.13)$$

$$\langle \mathcal{P}(w, u_a) | w \rangle_p \leq \| \mathcal{P}(w, u_a) \|_p \| w \|_p \leq K_p \| u_a \|_{p+1} \| w \|_p^2 \leq K_p \mathcal{D}_{p+1} \| w \|_p^2, \quad (4.14)$$

$$\langle \mathcal{P}(w, w) | w \rangle_p \leq G_{pn} \| w \|_n \| w \|_p^2, \quad (4.15)$$

$$- \langle e(u_a) | w \rangle_p \leq \| e(u_a) \|_p \| w \|_p \leq \epsilon_p \| w \|_p. \quad (4.16)$$

Inserting (4.12)-(4.16) into (4.11) one obtains the relation

$$\frac{d^+ \| w \|_p}{dt} \leq -\nu \| w \|_p + (G_p \mathcal{D}_p + K_p \mathcal{D}_{p+1}) \| w \|_p + G_{pn} \| w \|_n \| w \|_p + \epsilon_p; \quad (4.17)$$

this is just the thesis (4.7), in a neighborhood of the instant t under consideration for which we were assuming $w(t) \neq 0$.

To conclude, we show that Eq.(4.17) holds as well at any instant t such that $w(t) = 0$. In fact, at any such instant we have

$$\frac{d^+ \| w \|_p}{dt} \leq_1 \left\| \frac{dw}{dt} \right\|_p =_2 \| e(u_a) \|_p \leq \epsilon_p =_3 \text{ r.h.s. of (4.7)}. \quad (4.18)$$

In the above, the inequality \leq_1 follows from a general property of the Dini derivative (see, e.g., [22]); the equality $=_2$ follows from (4.9) and from $w(t) = 0$; the equality $=_3$ uses again the vanishing of w at this instant. \square

4.3 Remark. The inequality (4.7) was presented in our work [14] in the special case $p = n$ (and in a different framework reviewed in the Introduction, where u, u_a were just continuous as maps to $\mathbb{H}_{\Sigma_0}^{n+2}$ and C^1 as maps to $\mathbb{H}_{\Sigma_0}^n$); in this work we pointed out the relations between this ($p = n$) inequality and a similar result obtained in [2], to which [14] is greatly indebted. The proof of Lemma 4.2 combines ideas from the cited works with the tame generalization (2.10) of the Kato inequality.

We are now ready to state our main result.

4.4 Proposition. *Consider a real $n > d/2 + 1$, and assume there is a function $\mathcal{R}_n \in C([0, T_c], \mathbf{R})$, with $T_c \in (0, T_a]$, fulfilling the following control inequalities:*

$$\frac{d^+ \mathcal{R}_n}{dt} \geq -\nu \mathcal{R}_n + (G_n \mathcal{D}_n + K_n \mathcal{D}_{n+1}) \mathcal{R}_n + G_n \mathcal{R}_n^2 + \epsilon_n \text{ everywhere on } [0, T_c), \quad (4.19)$$

$$\mathcal{R}_n(0) \geq \delta_n \quad (4.20)$$

(note that (4.19) (4.20) are fulfilled as equalities by a unique function in $C^1([0, T_c], \mathbf{R})$, for a suitable T_c). Then, (i)(ii) hold.

(i) The maximal solution u of the NS Cauchy problem and its time of existence T are such that

$$T \geq T_c, \quad (4.21)$$

$$\|u(t) - u_{\mathbf{a}}(t)\|_n \leq \mathcal{R}_n(t) \quad \text{for } t \in [0, T_c) . \quad (4.22)$$

In particular, if \mathcal{R}_n is global ($T_c = +\infty$) then u is global as well ($T = +\infty$).

(ii) Consider any real $p > n$, and let $\mathcal{R}_p \in C([0, T_c), \mathbf{R})$ be a solution of the linear control inequalities

$$\frac{d^+ \mathcal{R}_p}{dt} \geq -\nu \mathcal{R}_p + (G_p \mathcal{D}_p + K_p \mathcal{D}_{p+1} + G_{pn} \mathcal{R}_n) \mathcal{R}_p + \epsilon_p \quad \text{everywhere on } [0, T_c) , \quad (4.23)$$

$$\mathcal{R}_p(0) \geq \delta_p . \quad (4.24)$$

Then

$$\|u(t) - u_{\mathbf{a}}(t)\|_p \leq \mathcal{R}_p(t) \quad \text{for } t \in [0, T_c) . \quad (4.25)$$

Conditions (4.23) (4.24) are fulfilled as equalities by a unique function $\mathcal{R}_p \in C^1([0, T_c), \mathbf{R})$, given explicitly by

$$\mathcal{R}_p(t) = e^{-\nu t} + \mathcal{A}_p(t) \left(\delta_p + \int_0^t ds e^{\nu s} - \mathcal{A}_p(s) \epsilon_p(s) \right) \quad \text{for } t \in [0, T_c) , \quad (4.26)$$

$$\mathcal{A}_p(t) := \int_0^t ds (G_p \mathcal{D}_p(s) + K_p \mathcal{D}_{p+1}(s) + G_{pn} \mathcal{R}_n(s)) . \quad (4.27)$$

□

Proof. (i) We use the inequality (4.7) of Lemma 4.2 with $p = n$, so that $G_{pn} = G_n$; this inequality reads

$$\frac{d^+}{dt} \|u - u_{\mathbf{a}}\|_n \leq -\nu \|u - u_{\mathbf{a}}\|_n \quad (4.28)$$

$$+(G_n \mathcal{D}_n + K_n \mathcal{D}_{n+1}) \|u - u_{\mathbf{a}}\|_n + G_n \|u - u_{\mathbf{a}}\|_n^2 + \epsilon_n \quad \text{on } [0, \min(T, T_{\mathbf{a}})) .$$

Moreover, by the very definition of the estimator δ_n we have

$$\|u(0) - u_{\mathbf{a}}(0)\|_n \leq \delta_n . \quad (4.29)$$

The inequalities (4.28) (4.29) for $\|u - u_{\mathbf{a}}\|_n$ have the same structure as the control inequalities (4.19) (4.20) for \mathcal{R}_n , with the reverse order relations; now, a standard comparison theorem à la Čaplygin-Lakshmikantham [9] [11] gives

$$\|u(t) - u_{\mathbf{a}}(t)\|_n \leq \mathcal{R}_n(t) \quad \text{for } t \in [0, \min(T, T_{\mathbf{a}}, T_c)) = [0, \min(T, T_c)) . \quad (4.30)$$

Finally, one has

$$\min(T, T_c) = T_c ; \quad (4.31)$$

in fact, if $T < T_c$, for all $t \in [0, T)$ we would have $\|u(t)\|_n \leq \|u(t) - u_{\mathbf{a}}(t)\|_n + \|u_{\mathbf{a}}(t)\|_n \leq \mathcal{R}_n(t) + \mathcal{D}_n(t)$ and this would imply $\limsup_{t \rightarrow T^-} \|u(t)\|_n \leq \mathcal{R}_n(T) + \mathcal{D}_n(T) < +\infty$, contradicting item (iii) of Proposition 3.2.

(ii) Keeping in mind item (i) we consider the inequality (4.7) for $\|u - u_{\mathbf{a}}\|_p$, holding on $[0, \min(T, T_{\mathbf{a}}))$ and, *a fortiori*, on the shorter interval $[0, T_c)$; from (4.7) and from $\|u - u_{\mathbf{a}}\|_n \leq \mathcal{R}_n$ we get

$$\frac{d^+}{dt} \|u - u_{\mathbf{a}}\|_p \leq -\nu \|u - u_{\mathbf{a}}\|_p \quad (4.32)$$

$$+(G_p \mathcal{D}_p + K_p \mathcal{D}_{p+1} + G_{pn} \mathcal{R}_n) \|u - u_{\mathbf{a}}\|_p + \epsilon_p \quad \text{on } [0, T_c) .$$

We add to this the relation (4.4) $\|u(0) - u_{\mathbf{a}}(0)\|_p \leq \delta_p$. The inequalities (4.32) (4.4) for $\|u - u_{\mathbf{a}}\|_p$ have the same structure as the inequalities (4.23) (4.24) assumed for \mathcal{R}_p , with the reverse order relations; again, a comparison theorem *à la* Čaplygin-Lakshmikantham gives

$$\|u(t) - u_{\mathbf{a}}(t)\|_p \leq \mathcal{R}_p(t) \quad \text{for } t \in [0, T_c) . \quad (4.33)$$

Finally, one checks by elementary means that the function \mathcal{R}_p defined by (4.26)(4.27) is the unique C^1 function on $[0, T_c)$ fulfilling conditions (4.23)(4.24) as equalities. \square

4.5 Remark. Given \mathcal{R}_n , the linearity of the control inequalities (4.23) for the functions \mathcal{R}_p ($p > n$) is closely related to the linearity of the inequality (4.7) with respect to $\|u - u_{\mathbf{a}}\|_p$; on the other hand, this feature of (4.7) depends essentially on the tame structure of the generalized Kato inequality (2.10).

5 A simple application of the previous results

The forthcoming result is an application of Proposition 4.4 in which $u_{\mathbf{a}}$ is chosen to be zero at all times (and the forcing is assumed to vanish, just for simplicity; on this point, see the forthcoming Remark 5.2(ii)). For $\nu, t \in [0, +\infty)$, let us define

$$e_{\nu}(t) := \begin{cases} \frac{1 - e^{-\nu t}}{\nu} & \text{if } \nu > 0, \\ t & \text{if } \nu = 0 \end{cases} \quad (5.1)$$

(noting that $\lim_{\nu \rightarrow 0^+} \frac{1 - e^{-\nu t}}{\nu} = t$). In the sequel G_n and G_{pn} have the usual meaning, see Eqs. (2.8)(2.10); recall that $G_{pn} = G_n$ for $p = n$.

5.1 Proposition. *Consider the Cauchy problem (3.2) for the NS equations with viscosity ν , zero forcing ($f(t) = 0$ for all $t \geq 0$) and any datum $u_0 \in \mathbb{H}_{\Sigma 0}^{\infty}$; let*

$u \in C^\infty([0, T], \mathbb{H}_{\Sigma_0}^\infty)$ denote its maximal solution. After fixing a real $n > d/2 + 1$, let us put

$$T_c := \begin{cases} +\infty & \text{if } \nu > 0, \|u_0\|_n \leq \nu/G_n, \\ -\frac{1}{\nu} \log \left(1 - \frac{\nu}{G_n \|u_0\|_n} \right) & \text{if } \nu > 0, \|u_0\|_n > \nu/G_n, \\ \frac{1}{G_n \|u_0\|_n} & \text{if } \nu = 0 \end{cases} \quad (5.2)$$

(intending $1/(G_n \|u_0\|_n) := +\infty$ if $u_0 = 0$). Then, u and its interval of existence fulfill

$$T \geq T_c, \quad (5.3)$$

$$\|u(t)\|_p \leq \frac{\|u_0\|_p e^{-\nu t}}{\left[1 - G_n \|u_0\|_n e_\nu(t) \right]^{G_{pn}/G_n}} \quad \text{for all real } p \geq n, t \in [0, T_c]. \quad (5.4)$$

(So, if $\|u_0\|_n \leq \nu/G_n$, u is global ($T = +\infty$) and its norm of any Sobolev order decays exponentially for $t \rightarrow +\infty$.)

Proof. We consider for the Cauchy problem (3.2) the zero approximate solution

$$u_a(t) := 0 \quad \text{for all } t \in [0, +\infty). \quad (5.5)$$

The differential error of u_a is zero (since $f = 0$), and the datum error is $u_a(0) - u_0 = -u_0$; so, we have the error and growth estimators

$$\epsilon_p(t) := 0, \quad \delta_p := \|u_0\|_p, \quad \mathcal{D}_p(t) := 0 \quad (p \in \mathbf{R}). \quad (5.6)$$

After fixing a Sobolev order $n > d/2 + 1$, we consider the control inequalities (4.19) (4.20) corresponding to these estimators and try to fulfill them as equalities for an unknown function $\mathcal{R}_n \in C^1([0, T_c], \mathbf{R})$; in this way we are led to the Cauchy problem

$$\frac{d\mathcal{R}_n}{dt} = -\nu \mathcal{R}_n + G_n \mathcal{R}_n^2, \quad \mathcal{R}_n(0) = \|u_0\|_n. \quad (5.7)$$

The maximal solution has domain $[0, T_c)$ with T_c as in (5.2), and is given by

$$\mathcal{R}_n(t) := \frac{\|u_0\|_n e^{-\nu t}}{1 - G_n \|u_0\|_n e_\nu(t)} \quad \text{for } t \in [0, T_c). \quad (5.8)$$

According to item (i) of Proposition 4.4 we have $T \geq T_c$ and $\|u(t)\|_n \leq \mathcal{R}_n(t)$ on $[0, T_c)$; this justifies Eq. (5.3) and gives as well Eq. (5.4) for $p = n$ (since the right hand side of this equation equals $\mathcal{R}_n(t)$ when $p = n$). Now, let $p > n$. Item (ii) of Proposition 4.4 with the present estimators gives

$$\|u(t)\|_p \leq \mathcal{R}_p(t) \quad \text{for } t \in [0, T_c), \quad (5.9)$$

$$\mathcal{R}_p(t) := e^{-\nu t} + \mathcal{A}_p(t) \|u_0\|_p, \quad \mathcal{A}_p(t) := G_{pn} \int_0^t ds \mathcal{R}_n(s). \quad (5.10)$$

The computation of \mathcal{A}_p is elementary, and one concludes

$$\mathcal{R}_p(t) = \frac{\|u_0\|_p e^{-\nu t}}{\left[1 - G_n \|u_0\|_n e_\nu(t)\right]^{G_{pn}/G_n}}; \quad (5.11)$$

this result and (5.9) justify Eq. (5.4) for $p > n$. \square

5.2 Remarks. (i) Proposition 5.1 is an extension of Proposition 5.2 of [14], where the zero approximate solution was employed to discuss the NS Cauchy problem (with zero forcing) in finite order Sobolev spaces; in the cited paper we obtained Eq. (5.4) for the special case $p = n > d/2 + 1$.

(ii) Proposition 5.1 can be generalized to the case of nonzero forcing, with suitable assumptions on it. In this case, using the zero approximate solution we can again obtain explicit bounds on the interval of existence of the exact NS solution u and on its Sobolev norms; in particular, u is global if $\nu > 0$ and F_n , $\|u_0\|_n$ are sufficiently small for some $n > d/2 + 1$, where $F_n := \sup_{t \geq 0} \|f(t)\|_n$.

6 An outline of more sophisticated applications

The general framework for approximate NS solutions and control inequalities devised in [14] for a Sobolev setting of a given finite order has been employed in the same paper and in the related works [17–20] in a number of applications, typically in dimension $d = 3$, where the following situations have been considered.

- (a) $\nu \geq 0$ and u_a is a Galerkin approximant;
- (b) $\nu > 0$ and u_a is an expansion in powers of the “Reynolds number” $1/\nu$, truncated to some order N . More precisely $u_a(t) = \sum_{j=0}^N (1/\nu^j) u_{(j)}(\nu t)$, where the coefficients $u_{(j)}$ are determined requiring the differential error to be $O(1/\nu^N)$, and $u_a(0) = u_0$;
- (c) $\nu = 0$ and u_a is an expansion in powers of time, truncated to some order N . More precisely $u_a(t) = \sum_{j=0}^N t^j u_j$, where the coefficients u_j are determined requiring the differential error to be $O(t^N)$, and $u_a(0) = u_0$.

The strategy (a) or (b) gives a global solution for the control inequalities of [14] when ν is above some critical value $\nu_{cr} > 0$, depending on the initial datum; in this situation, one infers that the NS exact solution u is global as well.

The applications presented in the cited works typically involve initial data in $\mathbb{H}_{\Sigma_0}^\infty$ (such as the vortices of Behr-Nečas-Wu, see [14] [17–20], and the vortices of Taylor-Green and Kida-Murakami, see [20]); the forcing is often chosen to be zero, and could be assumed in any case to be in $C^\infty([0, +\infty), \mathbb{H}_{\Sigma_0}^\infty)$. Therefore, such applications can be reconsidered from the viewpoint of the present Proposition 4.4.

The control inequalities (4.19) (4.20) of a given Sobolev order $n > d/2 + 1$ appearing in this proposition are in fact identical to the ones of [14]; they have been already solved for the cited applications in that paper and in [17–20] (typically, for $d = n = 3$). From the viewpoint of the present Proposition 4.4, the existence of a solution $\mathcal{R}_n \in C([0, T_c), \mathbf{R})$ for these control inequalities (possibly, with $T_c = +\infty$) ensures that the NS Cauchy problem has a solution $u \in C^\infty([0, T), \mathbb{H}_{\Sigma_0}^\infty)$ with $T \geq T_c$, and that $\|u(t) - u_a(t)\|_n \leq \mathcal{R}_n(t)$ on $[0, T_c)$.

For each one of the cited applications, using the already known function \mathcal{R}_n with item (ii) of Proposition 4.4 we could estimate $\|u(t) - u_a(t)\|_p$ for $t \in [0, T_c)$ and an arbitrarily large Sobolev order p . Presenting here these implementations of item (ii) would bring the length of this paper above a reasonable bound; we plan to return on this subject in future works. These will refer to the numerical values of the constants G_{pn} appearing in item (ii) of Proposition 4.4, obtained on the grounds of Appendix A and [21].

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A Appendix. On the constants K_{pn}, G_{pn} in Eqs. (2.9) (2.10)

The constants K_n, G_n in Eqs. (2.7) (2.8) were estimated in [15] [16]. In [21] the approach of these papers is extended to the “tame” inequalities (2.9) (2.10), and it is shown that the constants therein can be taken as follows:

$$K_{pn} = \frac{1}{(2\pi)^{d/2}} \sqrt{\sup_{k \in \mathbf{Z}^d \setminus \{0\}} \mathcal{K}_{pn}(k)} , \quad G_{pn} = \frac{1}{(2\pi)^{d/2}} \sqrt{\sup_{k \in \mathbf{Z}^d \setminus \{0\}} \mathcal{G}_{pn}(k)} , \quad (\text{A.1})$$

where $\mathcal{K}_{pn}, \mathcal{G}_{pn} : \mathbf{Z}^d \setminus \{0\} \rightarrow (0, +\infty)$ are defined by

$$\mathcal{K}_{pn}(k) := 4|k|^{2p} \sum_{h \in \mathbf{Z}^d \setminus \{0, k\}} \frac{C_{h,k}^2}{(|h|^p |k - h|^{n+1} + |h|^n |k - h|^{p+1})^2} , \quad (\text{A.2})$$

$$\mathcal{G}_{pn}(k) := 4 \sum_{h \in \mathbf{Z}^d \setminus \{0, k\}} \frac{(|k|^p - |k - h|^p)^2 C_{h,k}^2}{(|h|^p |k - h|^n + |h|^n |k - h|^p)^2} . \quad (\text{A.3})$$

The coefficient $C_{h,k}$ in the above formulas is any upper bound on the norm of the bilinear map $h^\perp \times (k - h)^\perp \rightarrow k^\perp$, $(a, b) \mapsto (k - h) \bullet a \mathfrak{L}_k b$, where \perp indicates the orthogonal complement in \mathbf{C}^d and \mathfrak{L}_k the projection of \mathbf{C}^d onto k^\perp ; one can take

$$C_{h,k} = \frac{|h \wedge k|}{|h|} = |k| \sin \theta(h, k) \quad (\text{A.4})$$

where \wedge indicates the exterior product (the usual vector product, if $d = 3$) and $\theta(h, k) \in [0, \pi]$ is the angle between h and k .

For more details, and for a description of the numerical procedures to compute $\mathcal{K}_{pn}, \mathcal{G}_{pn}$ and their sups, we refer to [21]. For $p = n$ and $C_{h,k}$ as in (A.4), the expressions (A.1) (A.2) (A.3) for K_{pn} and G_{pn} agree with the ones proposed in [15] [16] for K_n and G_n . Admittedly, we do not know if the constants determined by (A.1) (A.2) (A.3) are sharp.

B Appendix. On the NS Cauchy problem, and the proof of Proposition 3.2

In this Appendix we stipulate

$$\nu \in [0, +\infty), \quad \sigma := \begin{cases} 1 & \text{if } \nu = 0, \\ 2 & \text{if } \nu > 0. \end{cases} \quad (\text{B.1})$$

Our aim is to sketch a proof of Proposition 3.2 about the NS Cauchy problem in a framework based on $\mathbb{H}_{\Sigma_0}^\infty$. This relies on known “hard” results on the Cauchy problem in finite order Sobolev spaces, summarized hereafter.

B.1 Definition. *Let*

$$p \in \mathbf{R}, p > d/2 + 1, \quad f \in C([0, +\infty), \mathbb{H}_{\Sigma_0}^p), \quad u_0 \in \mathbb{H}_{\Sigma_0}^p. \quad (\text{B.2})$$

The (incompressible) NS Cauchy problem with viscosity ν , Sobolev order p , forcing f and initial datum u_0 is the following:

$$\text{Find } u \in C([0, T], \mathbb{H}_{\Sigma_0}^p) \cap C^1([0, T], \mathbb{H}_{\Sigma_0}^{p-\sigma}) \quad \text{such that} \quad (\text{B.3})$$

$$\frac{du}{dt} = \nu \Delta u + \mathcal{P}(u, u) + f, \quad u(0) = u_0$$

(with $T \in (0, +\infty]$, depending on u). If $\nu = 0$, this will also be called the “Euler Cauchy problem”.

B.2 Proposition. *With ν, σ, p, f, u_0 as in (B.1) (B.2), the following holds.*

(i) Problem (3.2) has a unique maximal (i.e., not extendable) solution, from now on indicated with u , with a suitable domain $[0, T)$. Every solution is a restriction of the maximal one.

(ii) (Beale-Kato-Majda blow up criterion.) Let u, T be as before. If $T < +\infty$, then $\int_0^T ds \|\text{rot } u(t)\|_{L^\infty} = +\infty$, whence $\limsup_{t \rightarrow T^-} \|\text{rot } u(t)\|_{L^\infty} = +\infty$.

(iii) The result (ii) implies the following: if $T < +\infty$, then for each real n such that $d/2 + 1 < n \leq p$ one has $\int_0^T dt \|u(t)\|_n = +\infty$, whence $\limsup_{t \rightarrow T^-} \|u(t)\|_n = +\infty$.

Proof. (i) was proved by Kato in [6]; as for (ii), see the original Beale-Kato-Majda result [1] (with its extension by Kozono and Taniuchi [7] to an arbitrary space dimension d), or the book by Majda and Bertozzi [10]. (iii) follows noting that, by the Sobolev imbedding lemma, $\|\text{rot } u(t)\|_{L^\infty} \leq \text{constant} \|u(t)\|_n$ for all $t \in [0, T)$. \square

For completeness let us also mention that, prior to [1], Temam [24] proved in place of (ii) the following, slightly weaker result (ii’): if $T < +\infty$, then $\int_0^T dt \|\partial u(t)\|_{L^\infty} = +\infty$, where ∂u stands for the Jacobian matrix $(\partial_r u_s)$; (ii’) and the Sobolev lemma are sufficient to infer the previous statement (iii). (For $\nu > 0$ these blow-up criteria could be enriched with the Giga criterion [3] [8] $\int_0^T \|u(t)\|_{L^\infty}^2 < +\infty$, implying $\int_0^T dt \|u(t)\|_n^2 = +\infty$ for $d/2 < n \leq p$.)

Using Proposition B.2, it is easy to derive Proposition 3.2 about NS Cauchy problem in $\mathbb{H}_{\Sigma_0}^\infty$; the argument employed hereafter is very similar to one proposed by Temam [24] in the particular case of the Euler equations.

Proof of Proposition 3.2. Recall the assumption (3.1) $f \in C^\infty([0, +\infty), \mathbb{H}_{\Sigma_0}^\infty)$, $u_0 \in \mathbb{H}_{\Sigma_0}^\infty$.

Step 1. The notations $(B.3)_p$, u_p, T_p . In the sequel we need to consider the Cauchy problem (B.3) for many choices of the Sobolev order $p > d/2 + 1$. To avoid confusion, we indicate with $(B.3)_p$ this Cauchy problem and (provisionally) write u_p for its maximal solution, of domain $[0, T_p)$.

Step 2. Let $p, q > d/2 + 1$, with $p \leq q$. Then $T_q \leq T_p$ and $u_q = u_p \upharpoonright [0, T_q)$. In fact u_q is a solution of $(B.3)_q$, which implies that u_q is as well a solution of $(B.3)_p$; on the other hand, any solution of $(B.3)_p$ is a restriction of u_p .

Step 3. Let $p, q > d/2 + 1$. Then $T_q = T_p$ and $u_q = u_p$. It suffices to prove this for $p \leq q$. In this case we have the result of Step 2, and we must just show that $T_q = T_p$. Assume this does not hold; then $T_q < T_p$ due to Step 2 and, in particular, $T_q < +\infty$. So, by item (iii) of Proposition B.2 (with p, n replaced by q, p) we have $\limsup_{t \rightarrow T_q^-} \|u_q(t)\|_p = +\infty$. On the other hand, the result of Step 2 implies $\limsup_{t \rightarrow T_q^-} \|u_q(t)\|_p = \limsup_{t \rightarrow T_q^-} \|u_p(t)\|_p = \|u_p(T_q)\|_p < +\infty$. So we have a contradiction; the conclusion is $T_q = T_p$.

Step 4. The function u . We now denote with T the common value of T_p for all $p > d/2 + 1$, and with u the function u_p for any such p . For any $p > d/2 + 1$, u is continuous from $[0, T)$ to $\mathbb{H}_{\Sigma_0}^p$; this implies $u \in C([0, T), \mathbb{H}_{\Sigma_0}^\infty)$.

Step 5. $u \in C^\infty([0, T), \mathbb{H}_{\Sigma_0}^\infty)$, and u is a solution of Cauchy problem (3.2). For any $p > d/2 + 1$, the function $u = u_p$ is in $C^1([0, T), \mathbb{H}_{\Sigma_0}^{p-\sigma})$ and fulfills $du/dt = \nu \Delta u + \mathcal{P}(u, u) + f$ on $[0, T)$, $u(0) = u_0$. By the arbitrariness of p , we infer that $u \in C^1([0, T), \mathbb{H}_{\Sigma_0}^\infty)$ and u fulfills the previous differential equation in the framework of differential calculus in $\mathbb{H}_{\Sigma_0}^\infty$. But Δ is continuous from $\mathbb{H}_{\Sigma_0}^\infty$ to itself, \mathcal{P} is a continuous bilinear map from $\mathbb{H}_{\Sigma_0}^\infty \times \mathbb{H}_{\Sigma_0}^\infty$ to $\mathbb{H}_{\Sigma_0}^\infty$ and f is C^1 (in fact C^∞) from $[0, T)$ to $\mathbb{H}_{\Sigma_0}^\infty$; so, $du/dt = \nu \Delta u + \mathcal{P}(u, u) + f \in C^1([0, T), \mathbb{H}_{\Sigma_0}^\infty)$ and this implies $u \in C^2([0, T), \mathbb{H}_{\Sigma_0}^\infty)$. An iteration of this argument shows that $u \in C^k([0, T), \mathbb{H}_{\Sigma_0}^\infty)$ for each $k \in \mathbb{N}$, thus proving that $u \in C^\infty([0, T), \mathbb{H}_{\Sigma_0}^\infty)$.

Step 6. Let $u' \in C^\infty([0, T'), \mathbb{H}_{\Sigma_0}^\infty)$ be a solution of the Cauchy problem (3.2); then $T' \leq T$ and $u' = u \upharpoonright [0, T')$ (thus, u is the unique not extendable solution of (3.2)). In fact, for any $p > d/2 + 1$, u' is as well a solution of the Cauchy problem $(B.3)_p$; thus, by Proposition B.2 u' is a restriction of the maximal solution u_p of $(B.3)_p$, that coincides with u .

Step 7. Let $T < +\infty$; then $\int_0^T dt \|\text{rot} u(t)\|_{L^\infty} = +\infty$, whence $\limsup_{t \rightarrow T^-} \|\text{rot} u(t)\|_{L^\infty} = +\infty$. For each $n > d/2 + 1$ this implies $\int_0^T dt \|u(t)\|_n = +\infty$, whence $\limsup_{t \rightarrow T^-} \|u(t)\|_n = +\infty$. The statements on $\|\text{rot} u(t)\|_{L^\infty}$ follow choosing any $p > d/2 + 1$ and applying Proposition B.2 to the function $u_p = u$. The statements on $\|u(t)\|_n$ follow using again the Sobolev inequality. \square

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